# Semigroup $C^*$ Crossed Products and Toeplitz Algebras

Mamoon Ahmed
BSc (Yarmouk University, Jordan)
MSc (The University of Newcastle, Australia)

Thesis submitted to
the University of Newcastle, NSW
for the degree of
Doctor of Philosophy in Mathematics.

November, 2007

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

(Signed) ____________________

**Acknowledgements.**

To my supervisor, Associate Professor Alan Pryde. I owe many thanks for his friendship, wisdom and for helping me learn to write more clearly.

Thanks to the maths department at the University of Newcastle for the time I have spent here.

To Sean Vittadello, my office-mate, many thanks for helpful discussions and time we spent together.

To my parents and all my big family back home in Jordan, many many thanks for supporting and encouraging me all the time. I love you all and wish you a happy long life.

To my dear wife Noor, I owe more than I can write but I mention here her encouragement, support and patience. So many thanks for being with me.

And to my lovely kids Jude and Abdullah, I love you and thanks for being a good reason to continue my PhD studies.

# Contents

## Abstract

Let $(G, G_+)$ be a quasi-lattice-ordered group with positive cone $G_+$. Laca and Raeburn have shown that the universal $C^*$-algebra $C^*(G, G_+)$ introduced by Nica is a crossed product $B_{G_+} \times_\alpha G_+$ by a semigroup of endomorphisms. Subsequent research centered on totally ordered abelian groups. We generalize the results in [**2**], [**3**] and [**5**] to extend it to the case of discrete lattice-ordered abelian groups. In particular given a hereditary subsemigroup $H_+$ of $G_+$ we introduce a closed ideal $I_{H_+}$ of the $C^*$-algebra $B_{G_+}$. We construct an approximate identity for this ideal and show that $I_{H_+}$ is extendibly $\alpha$-invariant. It follows that there is an isomorphism between $C^*$-crossed products $(B_{G_+}/I_{H_+}) \times_{\widetilde{\alpha}} G_+$ and $B_{(G/H)_+} \times_\beta G_+$. This leads to one of our main results that $B_{(G/H)_+} \times_\beta G_+$ is realized as an induced $C^*$-algebra $\mathrm{Ind}_{H^\perp}^{\widehat{G}} \left(B_{(G/H)_+} \times_\tau (G/H)_+\right)$. Then we use this result to show the existence of the following short exact sequence of $C^*$-algebras

$$0 \to I_{H_+} \times_\alpha G_+ \to B_{G_+} \times_\alpha G_+ \to \mathrm{Ind}_{H^\perp}^{\widehat{G}} \left(B_{(G/H)_+} \times_\tau (G/H)_+\right) \to 0.$$

This leads to show that the ideal $I_{H_+} \times_\alpha G_+$ is generated by $\left\{i_{B_{G_+}}(1 - 1_u) : u \in H_+\right\}$ and therefore contained in the commutator ideal $\mathcal{C}_G$ of the $C^*$-algebra $B_{G_+} \times_\alpha G_+$. Moreover, we use our short exact sequence to study the primitive ideals of the $C^*$ algebra $B_{G_+} \times_\alpha G_+$ which is isomorphic to the Toeplitz algebra $\mathcal{T}(G)$ of $G$.