

Semigroup C^* Crossed Products and Toeplitz Algebras

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I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

(Signed) _____

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Contents

Abstract	8
Chapter 1. Introduction	9
Chapter 2. Preliminaries	15
2.1. Lattice and quasi-lattice ordered groups	15
2.2. Examples	18
Chapter 3. Semigroup dynamical systems and crossed products	23
3.1. Multiplier algebras and extendibility	23
3.2. Definition of the crossed product	24
3.3. Covariant isometric representations of \mathbb{N}^2	27
3.4. The C^* -subalgebra B_{G_+} of $\ell^\infty(G_+)$	28
Chapter 4. Extendibly invariant Ideals	35
4.1. Construction of the ideal I_{H_+}	35
4.2. The C^* -algebras $B_{(G/H)_+}$ and B_{G_+}/I_{H_+}	41
Chapter 5. Inflated dynamical systems	46
5.1. Irreducible representations of C^* -algebras	46
5.2. The relationship between inflated systems	47
5.3. Crossed products of inflated systems	49
5.4. The crossed product $B_{H_+} \times_\alpha H_+$ and its commutator ideal	60
Chapter 6. Primitive ideals in the crossed product $B_{G_+} \times_\alpha G_+$	66
6.1. Definitions and background material	66
6.2. The composition $Q \circ \widehat{\beta}_\gamma^{-1} \circ \theta_H$ and primitive ideals	67

6.3. Examples	70
Chapter 7. Some concluding remarks	75
Appendix A.	76
Appendix. Bibliography	79

Abstract

Let (G, G_+) be a quasi-lattice-ordered group with positive cone G_+ . Laca and Raeburn have shown that the universal C^* -algebra $C^*(G, G_+)$ introduced by Nica is a crossed product $B_{G_+} \times_\alpha G_+$ by a semigroup of endomorphisms. Subsequent research centered on totally ordered abelian groups. We generalize the results in [2], [3] and [5] to extend it to the case of discrete lattice-ordered abelian groups. In particular given a hereditary subsemigroup H_+ of G_+ we introduce a closed ideal I_{H_+} of the C^* -algebra B_{G_+} . We construct an approximate identity for this ideal and show that I_{H_+} is extendibly α -invariant. It follows that there is an isomorphism between C^* -crossed products $(B_{G_+}/I_{H_+}) \times_{\tilde{\alpha}} G_+$ and $B_{(G/H)_+} \times_\beta G_+$. This leads to one of our main results that $B_{(G/H)_+} \times_\beta G_+$ is realized as an induced C^* -algebra $\text{Ind}_{H^\perp}^{\hat{G}} (B_{(G/H)_+} \times_\tau (G/H)_+)$. Then we use this result to show the existence of the following short exact sequence of C^* -algebras

$$0 \rightarrow I_{H_+} \times_\alpha G_+ \rightarrow B_{G_+} \times_\alpha G_+ \rightarrow \text{Ind}_{H^\perp}^{\hat{G}} (B_{(G/H)_+} \times_\tau (G/H)_+) \rightarrow 0.$$

This leads to show that the ideal $I_{H_+} \times_\alpha G_+$ is generated by $\{i_{B_{G_+}}(1 - 1_u) : u \in H_+\}$ and therefore contained in the commutator ideal \mathcal{C}_G of the C^* -algebra $B_{G_+} \times_\alpha G_+$. Moreover, we use our short exact sequence to study the primitive ideals of the C^* algebra $B_{G_+} \times_\alpha G_+$ which is isomorphic to the Toeplitz algebra $\mathcal{T}(G)$ of G .